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Reinterpretation and Extension of Bell's Inequality for Multivalued Observables

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Abstract

Bell's inequality, as it is generally understood, is a relation involving average values over the readings of the apparatuses when dicotomic observables are measured, which can be derived on the basis of local hidden variables and can be violated by quantum mechanics. We derive a Bell type inequality, involving average values of physical quantities whose quantum mechanical operators need not be two-valued, under the hypothesis that composite systems are described by "proper" mixtures (i.e., statistical mixtures in the ordinary sense). This inequality is analyzed in detail for the case of a $J = M = 0$ state decaying into subsystems I, II with any j and when average values of the quantity $(\vec{J}\cdot\hat{a})_{\rm I}\otimes(\vec{J}\cdot\hat{b})_{\rm II}$ are considered; it is then found that it cannot be violated if $j > \frac{1}{2}$ by the quantum mechanical description of the composite system in terms of a "second-kind state." A theorem is, however, established proving that in the general case suitable observables can be introduced for which a violation by quantum mechanics could be observed. This encourages the work in progress on more general situations and observables.

1. *Introduction*

Quantum mechanics (Q.M.), since its codification in the late twenties, has known such imposing successes in the interpretation of physical reality that the opponents of the standard or, as it was named, orthodox interpretation were deprived of any real possibility of pursuing their efforts towards alternative formulations. Typically, Von Neumann's theorem on hidden variables (H.V.)—though, as is now generally accepted,[1] did not meet the very problem it was aimed at—was taken as proof that no alternative interpretation could be developed that would agree with the known experimental data.

Only fairly recently—and after a feeling of dissatisfaction had spread among research workers, particularly in the field of high-energy physics, for the inadequacies of the theoretical approaches and the fragmentary pattern emerging from the data—has the debate on the fundamentals of Q.M. been reopened.

After Bohm in a pioneering work had proved that a nonlocal H.V. model could give the same results as Q.M. (Bohm, 1952), Bell observed that very simple local H.V. models could be conceived that did not meet the prerequisites of Von Neumann's theorem, thus showing that they were too restrictive (Bell, 1966); on the other hand, though it was true that no local H.V. theory could reproduce all the results of Q.M., Bell showed (Bell, 1965), for dicotomic observables, that the quantum mechanical predictions concerning the class of experiments for which Q.M. could really be tested against local H.V. theories had never been adequately performed (see, for instance, Capasso et al., 1970).

It has been shown since (Capasso et al., 1973) that for dicotomic variables the observable differences are linked to the presence, in the quantum description, of second-kind states, i.e., states of a composite system (such as for instance a singlet state composed of two spin-$\frac{1}{2}$ states), which cannot be factored into pure states of the subsystems, as opposed to first-kind states. In other words, the quantum mechanical treatment of composite systems—which is certainly correct, at least as far as localized systems are concerned[2]—had never been seriously tested before the seventies in connection with other features.

Even more interesting was the circumstance that the early and crucial con-

[1] One can read in Belinfante's treatise (1973) on Hidden Variables:

> I always have been puzzled how people could ever have been convinced by von Neumann's arguments that hidden variables could not be introduced. The lack of validity of (31) [the crucial hypothesis on the linearity of the mean values] should have been obvious to anybody by inspection . . . The truth, however, happens to be that for decades nobody spoke up against von Neumann's arguments, and that his conclusions were quoted by some as the gospel.

[2] For instance, antisymmetric (second-kind) states are necessary for an understanding of the stability of matter in the light of the Pauli principle.

ceptual difficulties of Q.M.—like the Einstein–Podolski–Rosen (E.P.R.) paradox and the problem of measurement—all derived from the presence of second-kind states, as was more generally the case with every quantum mechanical feature that can be loosely speaking described as a nonlocal effect.

It has recently been argued (Capasso et al, 1973; Baracca et al., 1975) that in principle Bell's inequality actually allows one to discriminate between a description of composite systems by means of second-kind states (or, in terms of the ensemble on which the statistics of measurements is recorded, improper mixtures) or by means of proper mixtures (i.e., mixtures in the ordinary sense), rather than Q.M. and local H.V. theories. (A rederivation of this result for general observables with a discrete spectrum is given below.)

Of course, a description in terms of proper mixtures of first-kind states[3] is intrinsically local and non-quantum-mechanical; not every, and not only local H.V. theory need be formulated in terms of proper mixtures.

2. *First- and Second-Kind States: Mean Values*

Since the central point of our argument is the observable difference existing between the first- and second-kind states, we start with a recollection of the main definitions. We consider a system I + II composed (for simplicity) of two subsystems I and II. Let $\hat{M}^{\mathrm{I}}$, $\hat{N}^{\mathrm{II}}$ be two observables of the two subsystems, having, respectively, the eigenstates $|\phi_i\rangle$, $|\xi_j\rangle$

$$\begin{aligned} \hat{M}^{\mathrm{I}} | \phi_i \rangle &= m_i | \phi_i \rangle \\ \hat{N}^{\mathrm{II}} | \xi_j \rangle &= m_j | \xi_j \rangle \end{aligned} \tag{2.1}$$

The system I + II may be in two different physical situations. From the point of view of Q.M., if a state vector is assigned to the global system I + II, it must be written in the form

$$| \psi^{\mathrm{I+II}} \rangle = \sum_k w_k^{1/2} | \phi_{\lambda_k} \rangle | \xi_{\rho_k} \rangle \tag{2.2}$$

where [by a theorem established by Von Neumann (Von Neumann, 1932) and discussed in a previous paper (Baracca et al., 1974)] the rearrangement connected with the subscripts λ_k, ρ_k allows us to write a single sum with real coefficient, since $w_k \geqslant 0$. This is a state of the second kind according to the nomenclature previously introduced: Notice that in this situation neither of the two subsystems has a state vector, a feature that implies a "nonlocal" character and is the origin of the paradoxical aspects connected with Q.M.

A different situation is one in which we are given a collection of a great number of identical systems I + II, each being in one of the possible state vectors $|\phi_{\lambda_k}\rangle | \xi_{\rho_k}\rangle$ with an assigned weight, which we choose as w_k in order

[3] Proper mixtures of second-kind states (or mixtures of the third kind) are also of physical interest (G. Ghirardi, private communication).

to have the maximum similarity with case (2.2); such a situation is described by a density operator $\hat{U}^{\mathrm{I+II}}_{(\mathrm{PR})}$:

$$\hat{U}^{\mathrm{I+II}}_{(\mathrm{PR})} = \sum_k w_k \mid \phi_{\lambda_k}\rangle \mid \xi_{\rho_k}\rangle\langle \xi_{\rho_k} \mid \langle \phi_{\lambda_k} \mid \tag{2.3}$$

and is called a "proper" mixture (of course, of first-kind states).[4] It has lost the "nonlocal" features of case (2.2).

A density operator may obviously be written for the state (2.2):

$$\hat{U}^{\mathrm{I+II}}_{(\mathrm{IM})} = \mid \psi^{\mathrm{I+II}}\rangle\langle \psi^{\mathrm{I+II}} \mid = \sum_{k,k'} w_k^{1/2} w_{k'}^{1/2} \mid \phi_{\lambda_k}\rangle \mid \xi_{\rho_k}\rangle\langle \xi_{\rho_{k'}} \mid \phi_{\lambda_{k'}}\rangle \tag{2.4}$$

which differs from (2.3) for the presence of nondiagonal terms.

We shall be concerned with the mean values $\langle \hat{A}^{\mathrm{I}}\hat{B}^{\mathrm{II}}\rangle$ of operators, defined in the overall Hilbert space, of the form $\hat{A}^{\mathrm{I}} \otimes \hat{B}^{\mathrm{II}}$ for which we have

$$\langle \hat{A}^{\mathrm{I}}\hat{B}^{\mathrm{II}}\rangle = \mathrm{Tr}\,(\hat{U}^{\mathrm{I+II}}\hat{A}^{\mathrm{I}}\hat{B}^{\mathrm{II}}) \tag{2.5}$$

We must distinguish the following two cases (Baracca et al., 1974):

2.1. *Proper Mixture (State Vector of the First Kind):* If a_i, b_j are the eigenvalues of the two observables, one gets, using equation (2.3),

$$\begin{aligned}\langle \hat{A}^{\mathrm{I}}\hat{B}^{\mathrm{II}}\rangle &= \sum_{i,j} a_i, b_j \sum_k w_k \mid \langle a_i \mid \phi_{\lambda_k}\rangle \mid^2 \mid \langle b_j \mid \xi_{\rho_k}\rangle \mid^2 \\ &= \sum_k w_k \sum_i a_i \mid \langle a_i \mid \phi_{\lambda_k}\rangle \mid^2 \sum_j b_j \mid \langle b_j \mid \xi_{\rho_k}\rangle \mid^2 \\ &= \sum_k w_k \langle A_k\rangle\langle B_k\rangle\end{aligned} \tag{2.6}$$

where $\langle A_k\rangle$, $\langle B_k\rangle$ are the mean values of $\hat{A}^{\mathrm{I}}$, $\hat{B}^{\mathrm{II}}$ when the system I + II is in the state $\mid \psi_k\rangle = \mid \phi_{\lambda_k}\rangle \mid \xi_{\rho_k}\rangle$.

2.2 *Improper Mixture (State Vector of the Second Kind).* One gets, using equation (2.4),

$$\begin{aligned}\langle \hat{A}^{\mathrm{I}}\hat{B}^{\mathrm{II}}\rangle &= \langle \psi \mid \hat{A}^{\mathrm{I}} \otimes \hat{B}^{\mathrm{II}} \mid \psi\rangle \\ &= \sum_{i,j} a_i b_j \mid \sum_k w_k^{1/2} \langle \phi_{\lambda_k} \mid a_i\rangle\langle \xi_{\rho_k} \mid b_j\rangle \mid^2 \\ &= \sum_k w_k \langle A_k\rangle\langle B_k\rangle + \sum_{\substack{k,k' \\ (k \neq k')}} I_{k,k'}\end{aligned} \tag{2.7}$$

[4] Once it is recognized that $P_k = \mid \phi_{\lambda_k}\rangle\langle \phi_{\lambda_k} \mid$ and $Q_k = \mid \xi_{\rho_k}\rangle\langle \xi_{\rho_k} \mid$ are the density operators describing the pure states $\mid \phi_{\lambda_k}\rangle$ and $\mid \xi_{\rho_k}\rangle$, respectively in subsystems I and II, (3) can be profitably compared with Jauch's (1971) expression for the density matrix describing his systems, $\int \rho(\alpha)\, P_\alpha\, Q_\alpha\, d\alpha$, that can be inferred from the context of his paper and can then be interpreted as an extension to continuous spectra.

where

$$I_{k,k'} = \sum_{i,j} w_k^{1/2} w_{k'}^{1/2} a_i \langle \phi_{\lambda_k} | a_i \rangle \langle a_i | \phi_{\lambda_{k'}} \rangle \, b_j \, \langle \xi_{\rho_k} | b_j \rangle \langle b_j | \xi_{\rho_{k'}} \rangle$$

that is

$$I_{k,k'} = w_k^{1/2} w_{k'}^{1/2} \langle \phi_{\lambda_k} | \hat{A} | \phi_{\lambda_{k'}} \rangle \langle \xi_{\rho_{k'}} | \hat{B} | \xi_{\rho_{k'}} \rangle \tag{2.8}$$

Expression (2.8) is the well-known interference term appearing for improper mixtures.

3. *Observable Difference between Proper and Improper Mixtures for Discrete Multivalued Observables*

Let us consider observables $\hat{A}^{\mathrm{I}}, \hat{B}^{\mathrm{II}}, \hat{C}^{\mathrm{II}}, \hat{D}^{\mathrm{I}}$ having a discrete spectrum such that

$$\begin{aligned} |\langle \hat{A}_k^{\mathrm{I}} \rangle| &\leqslant M^{\mathrm{I}}, \qquad |\langle \hat{D}_k^{\mathrm{I}} \rangle| \leqslant M^{\mathrm{I}} \\ |\langle \hat{B}_k^{\mathrm{II}} \rangle| &\leqslant M^{\mathrm{II}}, \qquad |\langle \hat{C}_k^{\mathrm{II}} \rangle| \leqslant M^{\mathrm{II}} \end{aligned} \tag{3.1}$$

and write

$$M^2 = M^{\mathrm{I}} \cdot M^{\mathrm{II}} \tag{3.2}$$

We start from the obvious relations

$$\sum_k w_k = 1$$

$$|\langle \hat{B}_k^{\mathrm{II}} \rangle - \langle \hat{C}_k^{\mathrm{II}} \rangle| + |\langle \hat{B}_k^{\mathrm{II}} \rangle + \langle \hat{C}_k^{\mathrm{II}} \rangle| \leqslant 2M^{\mathrm{II}}$$

They imply that

$$\sum_k w_k |\langle \hat{B}_k^{\mathrm{II}} \rangle - \langle \hat{C}_k^{\mathrm{II}} \rangle| + \sum_k w_k |\langle \hat{B}_k^{\mathrm{II}} \rangle + \langle \hat{C}_k^{\mathrm{II}} \rangle| \leqslant 2M^{\mathrm{II}}$$

This in turn implies that

$$\sum_k w_k |\langle \hat{A}_k^{\mathrm{I}} \rangle| \cdot |\langle \hat{B}_k^{\mathrm{II}} \rangle - \langle \hat{C}_k^{\mathrm{II}} \rangle| + \sum_\kappa w_k |\langle \hat{D}_k^{\mathrm{I}} \rangle| \cdot |\langle \hat{B}_k^{\mathrm{II}} \rangle + \langle \hat{C}_k^{\mathrm{II}} \rangle| \leqslant 2M^2$$

and finally:

$$\left| \sum_k w_k \langle \hat{A}_k^{\mathrm{I}} \rangle \{\langle \hat{B}_k^{\mathrm{II}} \rangle - \langle \hat{C}_k^{\mathrm{II}} \rangle\} \right| + \left| \sum_\kappa w_k \langle \hat{D}_k^{\mathrm{I}} \rangle \{\langle \hat{B}_k^{\mathrm{II}} \rangle + \langle \hat{C}_k^{\mathrm{II}} \rangle\} \right| \leqslant 2M^2$$

which may be written using equation (2.6), i.e., proper mixtures, as

$$|\langle \hat{A}^{\mathrm{I}} \hat{B}^{\mathrm{II}} \rangle - \langle \hat{A}^{\mathrm{I}} \hat{C}^{\mathrm{II}} \rangle| + |\langle \hat{D}^{\mathrm{I}} \hat{B}^{\mathrm{II}} \rangle + \langle \hat{D}^{\mathrm{I}} \hat{C}^{\mathrm{II}} \rangle| \leqslant 2M^2 \tag{3.3a}$$

and finally

$$|\langle \hat{A}^{\mathrm{I}} \hat{B}^{\mathrm{II}} \rangle - \langle \hat{A}^{\mathrm{I}} \hat{C}^{\mathrm{II}} \rangle| \pm \{\langle \hat{D}^{\mathrm{I}} \hat{B}^{\mathrm{II}} \rangle + \langle \hat{D}^{\mathrm{I}} \hat{C}^{\mathrm{II}} \rangle\} \leqslant 2M^2 \tag{3.3b}$$

This is a generalization of Bell's inequality, which is recovered when $M = 1$. We conclude that Bell's inequality does not necessarily imply the existence of local hidden variables, but only of "locality" in the sense of proper mixtures. In the following sections we show that it can in fact in principle be violated by second-kind states. This had already been shown by Capasso et al. (1973) for dicotomic observables.

In this paper we limit ourselves to considering states with vanishing total angular momentum; we then show that actually, if the mean value of the operator $(\mathrm{j}\cdot\hat{a})^{\mathrm{I}}\otimes(\mathrm{j}\cdot\hat{b})^{\mathrm{II}}$ in this state is considered, where $\hat{a}$ and $\hat{b}$ are any two directions, then inequality (3.3) can never be violated if the angular moments of the subsystems exceeds $\frac{1}{2}$; it can, however, be violated if particular couples of observables, whose physical meaning is analyzed, are considered.

The case of general $|J, M\rangle$ states will be considered in a forthcoming paper, together with an extension to observables described in general by vector operators.

Inequality (3.3) can of course also be derived on the basis of a local H.V. theory; it can then be considered as an extension to nondicotomic observables of the original Bell's inequality. Two points deserve some attention: In the first place, it must be noted that our mean values, such as $\langle\hat{A}\hat{B}\rangle$ are akin to the $P(a, b)$ of Bell's theory but not quite conceptually equivalent; the $P(a, b)$ arise in fact from averaging upon the readings of the apparatus (conventionally taken as ± 1); our mean values are in principle obtained as averages over observed values of the dynamical variables. This proves, as we have just seen, to be no obstacle to a treatment of the problem on the same footing. A further difficulty seems to arise when trying to derive the aforementioned generalization in terms of the $P(a, b)$, inasmuch as it is a widespread conviction that both the dicotomy and the limitation (by unity) of the amplitudes of the variables at hand play an essential role. As far as the latter point is concerned, it is quite trivially concluded that the above limiting values must be only a matter of scale, i.e., of units; as to the dicotomy, we need only stress that the recent derivations (Selleri, 1972, 1974) of Bell's inequality make no essential use of it (an explicit derivation is given in Appendix A).

4. *Second-Kind States (Improper Mixtures). Case A: Mean Values of Angular Momentum Correlations*

Once it is proved that Bell's inequality can be trivially extended to multivalued observables, it becomes of interest to evaluate the quantum mechanical values of the $\langle\hat{A}\hat{B}\rangle$ for cases of physical interest.

With reference to the E.P.R. (or Bohm–Aharonov, 1957) situation, the mean value of the observable $(\mathrm{j}\cdot\hat{a})\otimes(\mathrm{j}\cdot\hat{b})$ in the singlet state was considered. The most obvious generalization is obtained by considering the mean value of the observable $(\mathrm{j}\cdot\hat{a})^{\mathrm{I}}\otimes(\mathrm{j}\cdot\hat{b})^{\mathrm{II}}$ in a state $J = M = 0$: this could be the case of the γ decay of a spin $-$ 0 nucleus.[5] We shall then be concerned in this section with the evaluation of $\langle 00 | \mathbf{j}\cdot\hat{a}\mathbf{j}\cdot\hat{b} | 00 \rangle$ and the discussion of the result quoted in the preceding section concerning the impossibility of violating equation (3.3) for spins of the subsystems higher than $\frac{1}{2}$. One has in general

$$\langle 00 | \mathbf{j}\cdot\hat{a}\,\mathbf{j}\cdot\hat{b} | 00 \rangle = -\tfrac{1}{3}j(j+1)\cos\beta \tag{4.1}$$

where β is the angle between directions $\hat{a}$ and $\hat{b}$ and j is the angular momentum quantum number of either subsystem.

Vector Model Derivation of Equation (4.1). This result is most easily physically understood in terms of the vector model. In this model the state of vanishing total angular momentum is obtained with the two angular momentum vectors, of amplitude $\sqrt{j(j+1)}$, pointing in opposite directions; of course only $2j+1$ discrete orientations are allowed to each of them. If we let θ_{1a} be the angle formed by the direction of the first of the vectors with $\hat{a}$, the above orientations are determined by

$$\cos\theta_{1a}(m) = m/\sqrt{j(j+1)}$$

Now, we have, for such a state

$$\mathbf{j}\cdot\hat{a}\,\mathbf{j}\cdot\hat{b} = (\sqrt{j(j+1)}\cos\theta_{1a})(\sqrt{j(j+1)}\cos\theta_{2b})$$

where θ_{2b} is the angle formed by the direction of the second vector with respect to $\hat{b}$; alternatively, as $\theta_{2b} = \pi - \theta_{1a} - \beta$, if $\beta = \widehat{ab}$, we have

$$\mathbf{j}\cdot\hat{a}\,\mathbf{j}\cdot\hat{b} = j(j+1)(-\cos^2\theta_{1a}\cos\beta + \cos\theta_{1a}\sin\theta_{1a}\sin\beta)$$

The contribution of the odd term $\cos\theta_{1a}\sin\theta_{1a}$ averages out to zero and one is left with

$$\langle 00 | \mathbf{j}\cdot\hat{a}\,\mathbf{j}\cdot\hat{b} | 00 \rangle = j(j+1)\,\overline{(-\cos^2\theta_{1a})}\cos\beta$$

where

$$\overline{(-\cos^2\theta_{1a})} = \frac{1}{2j+1}\sum_{m=-j}^{j}\frac{-m^2}{j(j+1)} = -\tfrac{1}{3},$$

as

$$\sum_{m=1}^{j} m^2 = \frac{j(j+1)(2j+1)}{6}$$

Equation (4.1) follows. A more formal derivation of this result is given in Appendix B.

Once the above mean value is inserted, taking into account the minus sign and posing

$$\hat{A}^{\mathrm{I}} = (\mathbf{j}\cdot\hat{a})^{\mathrm{I}}, \qquad \hat{D}^{\mathrm{I}} = (\mathbf{j}\cdot\hat{d})^{\mathrm{I}}, \qquad \hat{B}^{\mathrm{II}} = (\mathbf{j}\cdot\hat{b})^{\mathrm{II}}, \qquad \hat{C}^{\mathrm{II}} = (\mathbf{j}\cdot\hat{c})^{\mathrm{II}},$$

[5] We thank P. Camiz for suggesting this example.

inequality (3.3b) takes the form

$$|\cos(\hat{ab}) - \cos(\hat{ac})| \mp \{\cos(\hat{db}) + \cos(\hat{dc})\} \leqslant 6j/(j+1)$$

We are interested in the maximum value of the left-hand side; one can easily see that it is obtained when the directions are coplanar and for the minus sign. As a consequence one is reduced to considering the inequality

$$|\cos\beta - \cos\beta'| - \cos(\alpha - \beta) - \cos(\alpha - \beta') \leqslant 6j/(j+1) \tag{4.2}$$

where β, β', α are respectively the angles $\hat{ab}$, $\hat{ac}$, $\hat{ad}$.

Then we can easily prove the following:

> *Theorem.* (a) The maximum value of the left-hand side of inequality (3.3) is $2\sqrt{2}$; (b) for improper mixtures [equation (2.4)], the mean values (4.1) of the "angular momentum correlations" cannot violate inequality (3.3) for $j > \frac{1}{2}$.

Proof: The maximum of the left-hand side is trivially computed by equating to zero the partial derivatives; notice that the expression is symmetric for the exchange $\beta \leftrightarrow \beta'$; it is then immediately observed that (a) the expression on the left-hand side has maxima at

$$\alpha = \frac{\pi}{2}, \qquad \beta' = \frac{5}{4}\pi, \qquad \beta = \frac{7}{4}\pi$$

$$\beta = \frac{\pi}{4}, \qquad \beta' = \frac{3}{4}\pi, \qquad \alpha = \frac{3}{2}\pi$$

which, substituted, give it the value $2\sqrt{2}$; (b) the right-hand side is obviously larger than $2\sqrt{2}$ for $j > \frac{1}{2}$. It is perhaps worth stressing that we are thus giving the prescription for a maximal violation.

5. *Second-Kind States (Improper Mixtures). Case B: Possible Choice of Observables that Violates the Generalized Bell's Inequality*

We intend to show that, for the $J = M = 0$ state, suitable observables can be chosen so as to violate inequality (3.3) even for spins higher than $\frac{1}{2}$.

Let $\{|\phi_i\rangle\}$ and $\{|\xi_i\rangle\}$ ($i = 1, 2$) be normalized vectors belonging to a basic set in the vector spaces for subsystems I and II, respectively, and let us introduce the projection operators[6]

$$\begin{aligned}
\hat{P}_{\mathrm{I}}^{(1)} &= |\phi_1\rangle\langle\phi_1| \\
\hat{P}_{\mathrm{II}}^{(1)} &= |\xi_1\rangle\langle\xi_1| \\
\hat{P}_{\mathrm{I}}^{(2)} &= [\alpha_1|\phi_1\rangle + \alpha_2|\phi_2\rangle]\,[\alpha_1^*\langle\phi_1| + \alpha_2^*\langle\phi_2|] \\
\hat{P}_{\mathrm{II}}^{(2)} &= [\beta_1|\xi_1\rangle + \beta_2|\xi_2\rangle]\,[\beta_1^*\langle\xi_1| + \beta_2^*\langle\xi_2|]
\end{aligned} \tag{5.1}$$

[6] These operators have been introduced by Capasso et al. (1973); we are here trying to make use of them for the case of nondicotomic observables.

We choose as our operators $\hat{A}, \hat{B}, \hat{C}, \hat{D}$ the following dicotomic observables:

$$\hat{D}^{(i)}_{\mathrm{I,II}} = 2\hat{P}^{(i)}_{\mathrm{I,II}} - 1, \qquad i = 1, 2 \tag{5.2}$$

having the eigenvalues ± 1 (we shall correspondingly refer to a Bell inequality with $M = 1$).

Both the projectors $\hat{P}$ and the $\hat{D}$ have an immediate physical meaning when subsystems I and II can be described by two-dimensional vector spaces; this could be the case, for instance, if I and II are spin-$\frac{1}{2}$ particles and we are interested only in internal degrees of freedom. In this case, if $\{|\phi_i\rangle\}$ and $\{|\xi_i\rangle\}$ are interpreted as eigenstates of the spin operator along a given direction for subsystems I and II, respectively, a state such as $\alpha_1|\phi_1\rangle + \alpha_2|\phi_2\rangle$ with $|\alpha_1|^2 + |\alpha_2|^2 = 1$ can be obtained through a rotation starting from $|\phi_1\rangle$ and correspondingly interpreted; consequently, the $\hat{P}$ are projectors onto the one-dimensional subspaces of given direction and the $\hat{D}$ are the corresponding spin observables.

In the case we have examined of angular momentum observables where $\{|\phi_i\rangle\}$ and $\{|\xi_i\rangle\}$ are interpreted as eigenstates of $\mathbf{J}^2, J_z$, the projectors (5.1) have no immediate physical meaning. This, however, produces no essential difficulty, once the coefficients of the second-kind state describing the overall system have been experimentally determined, since it is easily seen that the mean values of the external products of the $\hat{D}$ operators are determined by the expressions

$$\begin{aligned}
&\langle \hat{D}^{(1)}_{\mathrm{I}} \otimes \hat{D}^{(1)}_{\mathrm{II}} \rangle = 1 \\
&\langle \hat{D}^{(1)}_{\mathrm{I}} \otimes \hat{D}^{(2)}_{\mathrm{II}} \rangle = |\beta_1|^2 - 2|\beta_2|^2(w_1 + w_2 - \tfrac{1}{2}) \\
&\langle \hat{D}^{(2)}_{\mathrm{I}} \otimes \hat{D}^{(1)}_{\mathrm{II}} \rangle = |\alpha_1|^2 - 2|\alpha_2|^2(w_1 + w_2 - \tfrac{1}{2}) \\
&\langle \hat{D}^{(2)}_{\mathrm{I}} \otimes \hat{D}^{(2)}_{\mathrm{II}} \rangle = (w_1 + w_2)\,\{(|\alpha_1|^2 - |\alpha_2|^2)(|\beta_1|^2 - |\beta_2|^2) - 1\} \\
&\qquad\qquad + 1 + 8\,\mathrm{Re}\,(w_1^{1/2} w_2^{1/2} \alpha_1^* \alpha_2 \beta_1^* \beta_2)
\end{aligned} \tag{5.3}$$

If $w_1 + w_2 = 1$ one recovers the results of Capasso et al. (1973).[7] In terms of these observables the inequality (3.3) takes the form

$$|\langle \hat{D}^{(1)}_{\mathrm{I}} \otimes \hat{D}^{(1)}_{\mathrm{II}} \rangle - \langle \hat{D}^{(1)}_{\mathrm{I}} \otimes \hat{D}^{(2)}_{\mathrm{II}} \rangle| + |\langle \hat{D}^{(2)}_{\mathrm{I}} \otimes \hat{D}^{(1)}_{\mathrm{II}} \rangle + \langle \hat{D}^{(2)}_{\mathrm{I}} \otimes \hat{D}^{(2)}_{\mathrm{II}} \rangle| \leqslant 2 \tag{5.4}$$

Then we can prove the following:

> *Theorem.* For any improper mixture, the observables $\hat{D}$ can always be chosen in such a way that the inequality (5.4) is not satisfied (if no superselection rules are present).

Proof. We strictly follow the procedure of Capasso et al. (1973), retaining that in general $w_1 + w_2 \leqslant 1$. Using the observables $\hat{D}^i$ the inequality

[7] Notice, however, (Baracca et al., 1974) that the coefficients w_k are here real, while they are taken as complex by Capasso et al. (1973).

takes the form ($\Delta\alpha = |\alpha_1|^2 - |\alpha_2|^2$, $\Delta\beta = |\beta_1|^2 - |\beta_2|^2$)

$$\begin{aligned}&||\alpha_1|^2 - 2|\alpha_2|^2[w_1 + w_2 - \tfrac{1}{2}] + 1 + \Delta\alpha\Delta\beta(w_1 + w_2) - w_1 + w_2 \\ &\quad + 8\,\mathrm{Re}\,(w_1^{1/2}w_2^{1/2}\alpha_1^*\alpha_2\beta_1^*\beta_2)| + |1 - |\beta_1|^2 + 2|\beta_2|^2[w_1 + w_2 - \tfrac{1}{2}]| \leqslant 2\end{aligned} \tag{5.5}$$

We write

$$\begin{aligned}8\,\mathrm{Re}\,(w_1^{1/2}w_2^{1/2}\alpha_1^*\alpha_2\beta_1^*\beta_2) &= 8\sqrt{w_1 w_2}\,\mathrm{Re}\,(\alpha_1^*\alpha_2\beta_1^*\beta_2) \\ &= 8\sqrt{w_1 w_2}\,|\alpha_1||\alpha_2||\beta_1||\beta_2|\,\mathrm{Re}\,e^{i(\varphi_{\alpha_1} - \varphi_{\alpha_2} + \varphi_{\beta_1} - \varphi_{\beta_2})} \\ &= 8\sqrt{w_1 w_2}\,|\alpha_1||\alpha_2||\beta_1||\beta_2|\cos(\Phi_a + \Phi_b)\end{aligned}$$

where the φ_i are the arguments of the complex numbers α_i, β_i with obvious notations, and $\Phi_a = \varphi_{\alpha_1} - \varphi_{\alpha_2}$, $\Phi_b = \varphi_{\beta_1} - \varphi_{\beta_2}$. Recalling that $|\alpha_1|^2 + |\alpha_2|^2 = |\beta_1|^2 + |\beta_2|^2 = 1$ one gets easily

$$\begin{aligned}&|2 - 2|\alpha_2|^2(w_1 + w_2) + \Delta\alpha\Delta\beta(w_1 + w_2) - (w_2 + w_2) \\ &+ 2\sqrt{w_1 w_2}\,[(1 - \Delta\alpha^2)(1 - \Delta\beta^2)]^{1/2}\cos(\Phi_a + \Phi_b)| + 2|\beta_2|^2(w_1 + w_2) \leqslant 2\end{aligned}$$

This relation is *certainly violated* if the other one without the moduli and with $\cos(\Phi_a + \Phi_b) = +1$ is violated, that is, if the following relation holds:

$$\begin{aligned}&2 - 2|\alpha_2|^2(w_1 + w_2) - (w_1 + w_2) + \Delta\alpha\Delta\beta(w_1 + w_2) \\ &\quad + 2\sqrt{w_1 w_2}\,[(1 - \Delta\alpha^2)(1 - \Delta\beta^2)]^{1/2} + 2|\beta_2|^2(w_1 + w_2) \leqslant 2\end{aligned} \tag{5.6}$$

This relation may easily be written as

$$\frac{2\sqrt{w_1 w_2}}{w_1 + w_2}[(1 - \Delta\alpha^2)(1 - \Delta\beta^2)]^{1/2} - 2|\alpha_2|^2 - \Delta\beta(1 - \Delta\alpha) \leqslant 0$$

and this may be violated if

$$\frac{2\sqrt{w_1 w_2}}{w_1 + w_2}[(1 - \Delta\alpha^2)(1 - \Delta\beta^2)]^{1/2} > (1 + \Delta\beta)(1 - \Delta\alpha)$$

We conclude that the original inequality (5.5) is certainly violated when

$$\left(\frac{1 + \Delta\alpha}{1 - \Delta\alpha}\,\frac{1 - \Delta\beta}{1 + \Delta\beta}\right)^{1/2} > \frac{w_1 + w_2}{2\sqrt{w_1 \cdot w_2}} \tag{5.7}$$

This condition reduces to the one of Bell (1966) when $w_1 + w_2 = 1$, that is for dicotomic observables (bidimensional "Hilbert" space). One may easily verify that there are values of $\Delta\alpha$, $\Delta\beta$ that satisfy (5.7).

We may also evaluate the maximum value of the left-hand side of the inequality; if we rewrite the relation (5.6) as

$$2 + f(\Delta\alpha, \Delta\beta, w_1, w_2) \leqslant 2 \tag{5.8}$$

we may equate to zero the partial derivatives of f with respect to $\Delta\alpha$, $\Delta\beta$. We find the maximum value

$$\max f = \frac{4w_1w_2}{w_2 + w_2 + 2\sqrt{w_1w_2}} \tag{5.9}$$

when

$$\Delta\alpha = -\Delta\beta = \frac{w_1 + w_2}{w_1 + w_2 + 2\sqrt{w_1w_2}} \tag{5.10}$$

Through a tabulation of (5.9) we have found that the maximum violation of (5.8) occurs for the dicotomic case when

$$w_1 = w_2 = 0.5 \text{ (singlet or triplet } m = 0 \text{ cases)}$$
$$\max f = 0.5$$

when

$$\Delta\alpha = -\Delta\beta = 0.5$$

that is,

$$\begin{cases} |\alpha_2|^2 = \frac{3}{4}, & |\alpha_2|^2 = \frac{1}{4} \\ |\beta_1|^2 = \frac{1}{4}, & |\beta_2|^2 = \frac{3}{4} \end{cases}$$

It should be noted that we find a maximal violation of 0.5.

Fortunato (1974) finds a larger violation, for dicotomic observables, by generalizing the form of the projector $\hat{P}_{\text{II}}^{(1)}$:

$$\hat{P}_{\text{II}}^{(1)} = [\gamma_1|\xi_1\rangle + \gamma_2|\xi_2\rangle][\gamma_1^*\langle\xi_1| + \gamma_2^*\langle\xi_2|]$$

It does not seem very useful at the moment to analyze such a generalization for multivalued observables; we try, rather, in a work in progress, to generalize all these results to more general states and observables than those studied in this paper.

Appendix A

We give here a straightforward extension of the original Bell inequality to nondicotomic observables; the derivation follows step by step a procedure worked out by Selleri (1974), for the dicotomic case.

Let us consider two couples of dynamical variables $A^{\text{I}}(a, \lambda)$, $D^{\text{I}}(d, \lambda)$; $B^{\text{II}}(b, \lambda)$, $C^{\text{II}}(c, \lambda)$ (in Bell's notation), which are allowed to assume denumerable sets of values, limited, in some units, by M^{I} and M^{II}, respectively; the measurements will associate with these values equal numbers of readings of the two apparatuses; there is no difficulty in interpreting M^{I} and M^{II} as upper limits, in some scale, for the above readings. In either case we thus have [cf. equation (3.1)]

$$|A^{\mathrm{I}}(a,\lambda)|\leqslant M^{\mathrm{I}},\qquad |D^{\mathrm{I}}(d,\lambda)|\leqslant M^{\mathrm{I}}$$
$$|B^{\mathrm{II}}(b,\lambda)|\leqslant M^{\mathrm{II}},\qquad |C^{\mathrm{II}}(c,\lambda)|\leqslant M^{\mathrm{II}}$$

we then define, as usual,

$$P(a,b)=\int d\lambda\rho(\lambda)A^{\mathrm{I}}(a,\lambda)B^{\mathrm{II}}(b,\lambda)$$

where the $P(a, b)$ have definite meanings in both cases. Then

$$\begin{aligned}|P(a,b)-P(a,c)|&\leqslant\int d\lambda\rho(\lambda)\,|A^{\mathrm{I}}(a,\lambda)B^{\mathrm{II}}(b,\lambda)-A^{\mathrm{I}}(a,\lambda)C^{\mathrm{II}}(c,\lambda)|\\&=\int d\lambda\rho(\lambda)\,|A^{\mathrm{I}}(a,\lambda)|\cdot|B^{\mathrm{II}}(b,\lambda)-C^{\mathrm{II}}(c,\lambda)|\\&\leqslant M^{\mathrm{I}}\int d\lambda\rho(\lambda)\,|B^{\mathrm{II}}(b,\lambda)-C^{\mathrm{II}}(c,\lambda)|\end{aligned}$$

One has similarly

$$|P(d,b)+P(d,c)|\leqslant M^{\mathrm{I}}\int d\lambda\rho(\lambda)\,|B^{\mathrm{II}}(b,\lambda)+C^{\mathrm{II}}(c,\lambda)|$$

whence

$$\begin{aligned}&|P(a,b)-P(a,c)|+|P(d,b)+P(d,c)|\\&\leqslant M^{\mathrm{I}}\int d\lambda\rho(\lambda)\{|B^{\mathrm{II}}(b,\lambda)-C^{\mathrm{II}}(c,\lambda)|+|B^{\mathrm{II}}(b,\lambda)+C^{\mathrm{II}}(c,\lambda)|\}\end{aligned}$$

From the inequality

$$|B^{\mathrm{II}}(b,\lambda)-C^{\mathrm{II}}(c,\lambda)|+|B^{\mathrm{II}}(b,\lambda)+C(c,\lambda)|\leqslant 2M^{\mathrm{II}}$$

there follows, with $M^2 \doteq M^{\mathrm{I}}M^{\mathrm{II}}$,

$$|P(a,b)-P(a,c)|+|P(d,b)+P(d,c)|\leqslant 2M^2$$

Appendix B

We give here a more formal derivation of equation (4.1) of the text. Since we have for a $J = M = 0$ state

$$\begin{aligned}|jj00\rangle&=\sum_{m_1,m_2}|jm_1jm_2\rangle\langle jm_1jm_2|jj00\rangle\\&=\sum_m|jmj-m\rangle\langle jmj-m|jj00\rangle\end{aligned}$$

and

$$\langle jmj-m|00\rangle=(-1)^{j-m}/\sqrt{2j+1}$$

in general, we have

$$|jj00\rangle=\sum_m\frac{(-1)^{j-m}}{\sqrt{2j+1}}\,|jmj-m\rangle$$

Then

$$\langle 00 | \mathbf{j}\cdot\hat{a}\,\mathbf{j}\cdot\hat{b} | 00\rangle = \sum_{m,m'} \frac{(-1)^{2j-m-m'}}{2j+1} \langle jm'j-m' | \mathbf{j}\cdot\hat{a}\,\mathbf{j}\cdot\hat{b} | jmj-m\rangle$$

$$= \frac{1}{2j+1} \sum_{m,m'} (-1)^{2j-m-m'} \langle jm' | \mathbf{j}\cdot\hat{a} | jm\rangle\langle j-m' | \mathbf{j}\cdot\hat{b} | j-m\rangle$$

In a Cartesian frame x, y, z one has

$$\mathbf{j}\cdot\hat{a} = j_x \cos(\hat{xa}) + j_y \cos(\hat{ya}) + j_z \cos(\hat{za})$$

and similarly for $\mathbf{j}\cdot\hat{b}$.

We can fix the reference frame with the plane coinciding with the plane determined by directions $\hat{a}$ and $\hat{b}$ and with $\hat{x} \equiv \hat{a}$. Then we have

$$\cos(\hat{ya}) = \cos(\hat{yb}) = \cos(\hat{za}) = 0$$

and

$$\mathbf{j}\cdot\hat{a} = j_x$$

whereas

$$\mathbf{j}\cdot\hat{b} = j_x \cos(\hat{ab}) + j_z \cos(\hat{zb})$$

Hence, as terms in $j_x j_z$ do not contribute in the sum,

$$\langle 00 | \mathbf{j}\cdot\hat{a}\,\mathbf{j}\cdot\hat{b} | 00\rangle = \frac{1}{2j+1} \left\{ \sum_{m,m'} (-1)^{2j-m-m'} \langle jm' | j_x | jm\rangle \langle j-m' | j_x | j-m\rangle \right\} \cos(\hat{ab}) \qquad \text{(B1)}$$

Once the explicit expression for the matrix elements

$$\langle jm' | j_x | jm\rangle = \tfrac{1}{2}\delta_{m',m+1} [(j-m)(j+m+1)]^{1/2} + \tfrac{1}{2}\delta_{m',m-1} [(j+m)(j-m+1)]^{1/2}$$

is inserted, equation (B1) reduces to

$$\langle 00 | \mathbf{j}\cdot\hat{a}\,\mathbf{j}\cdot\hat{b} | 00\rangle = -\tfrac{1}{4} \frac{1}{2j+1} \cos(\hat{ab}) \times \sum_{m=-j}^{j} [(j-m)(j+m+1) + (j+m)(j-m+1)]$$

The sum over m is easily computed to give $4/3\, j(j+1)(2j+1)$ and equation (4.1) is recovered.

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